

HARDY-ORLICZ SPACES: TAYLOR COEFFICIENTS, INTERPOLATION OF OPERATORS, AND UNIVALENT FUNCTIONS

(Dedicated to the memory of Milutin Dostanić)

MIROSLAV PAVLOVIĆ

Faculty of Mathematics
University of Belgrade
Belgrade
Serbia
e-mail: miroslavpvlvc@gmail.com
pavlovic@matf.bg.ac.rs

Abstract

We present some inequalities for the Taylor coefficients of a Hardy-Orlicz function, which in the case of the standard Hardy spaces reduce to the inequalities of Hardy and Littlewood. The proofs are new and elementary even in the case of the classical Hardy spaces. We also prove a Marcinkiewicz type theorem that extends a theorem of Kislyakov and Xu. At the end, we prove a Hardy-Prawitz criterion for membership of a univalent function in a Hardy-Orlicz space.

1. Notation and Basic Facts

By the term “Orlicz function”, we mean a continuous function $\Phi : [0, \infty) \mapsto [0, \infty)$ such that $\Phi(0) = 0$ and

$$\frac{\Phi(t)}{t^\alpha} \text{ increases in } t \in (0, \infty), \text{ for some } \alpha > 0. \quad (1)$$

2010 Mathematics Subject Classification: 30H10, 46E30.

Keywords and phrases: Orlicz spaces, Hardy-Orlicz spaces, Taylor coefficients, interpolation of linear operators.

Received March 23, 2014

We shall also consider the following possible condition, called Δ_2 -condition:

$$\sup_{t>a} \frac{\Phi(2t)}{\Phi(t)} < \infty, \text{ for some constant } a \geq 0. \quad (2)$$

This condition is implied by

$$\frac{\Phi(t)}{t^\beta} \text{ decreases in } t \in (0, \infty), \text{ for some } \beta > 0. \quad (3)$$

If Φ satisfies both (1) and (3), then we write $\Phi \in \Delta[\alpha, \beta]$. We write $\Phi \in \Delta(\alpha, \beta)$, if $\Phi \in \Delta[\gamma, \delta]$ for some $\alpha < \gamma < \delta < \beta$.

Let μ be a positive finite measure defined “on a set S ”. If Φ is an Orlicz function, then the (Orlicz) space $L_\Phi(\mu)$ consists, by definition, of those complex valued functions¹ f such that

$$\int_S \Phi\left(\frac{|f(s)|}{\lambda}\right) d\mu(s) < \infty, \text{ for some } \lambda > 0. \quad (4)$$

The subspace of L_Φ defined by the requirement “for all $\lambda > 0$ ” is denoted by $E_\Phi(\mu)$. The Luxemburg (quasi) norm on L_Φ is defined to be the infimum of those λ for which (4) holds. The spaces L_Φ and E_Φ are complete (see, e.g., [7, page 36], or [4]).

If the measure μ is finite and there are positive constants c_j and C_j , and $a > 0$, such that

$$c_1 \Psi(c_2 t) \leq \Phi(t) \leq C_1 \Psi(C_2 t), \quad t \geq a, \quad (5)$$

where Φ and Ψ are Orlicz function, then $L_\Phi = L_\Psi$, $E_\Phi = E_\Psi$, with equivalent norms. We write $\Psi(t) \sim \Phi(t)$, $t \geq a$, if (5) holds.

¹We always assume that functions under consideration are measurable.

Lemma 1.1. *If Φ is an Orlicz function satisfying (1), then there is an Orlicz function Ψ such that*

- (i) $\Psi(t) \sim \Phi(t)$, $t \geq 0$;
- (ii) *the function $\Psi(t^{1/\alpha})$, $t \geq 0$ is convex.*

Proof. The desired conditions are satisfied by the function

$$\Psi(t) = \int_0^{t^\alpha} \frac{\Phi(x^{1/\alpha})}{x} dx, \quad t \geq 0.$$

Indeed, if (1) holds, then $\frac{\Phi(x^{1/\alpha})}{x}$ increases with x and so

$$\Psi(t^{1/\alpha}) \leq \int_0^t \frac{\Phi(t^{1/\alpha})}{t} dt = \Phi(t^{1/\alpha}), \quad t > 0.$$

On the other hand, since Φ is increasing, we have

$$\Psi(t^{1/\alpha}) \geq \int_{t/e}^t \frac{\Phi((t/e)^{1/\alpha})}{x} dx = \Phi((t/e)^{1/\alpha}),$$

and hence $\Psi(t) \geq \Phi(t/e^{1/\alpha})$, which completes the proof. □

Remark 1.2. We can consider a condition weaker than (1), namely:

$$\frac{\Phi(t)}{t^\alpha} \text{ almost increases in } t \in (0, \infty), \text{ for some } \alpha > 0. \tag{6}$$

According to Bernstein [1], a real function φ is said to be almost increasing if the implication “ $x < y \Rightarrow \varphi(x) \leq C\varphi(y)$ ” holds, where C is a positive constant independent of x and y . If Φ satisfies (6), then the function

$$\Psi(t) = t^\alpha \inf_{x \geq t} \frac{\Phi(x)}{x^\alpha}, \tag{7}$$

satisfies (1) (with Ψ instead of Φ), and

$$\frac{1}{C} \Phi(t) \leq \Psi(t) \leq \Phi(t), \quad t \geq 0. \tag{8}$$

Hardy-Orlicz spaces. If $S = \mathbb{T}$, the unit circle, and μ is the Haar measure on \mathbb{T} , then we write $L_\Phi(\mu) = L_\Phi(\mathbb{T})$ and $E_\Phi(\mu) = E_\Phi(\mathbb{T})$. The Hardy-Orlicz spaces $H_\Phi(\mathbb{T})$ and $EH_\Phi(\mathbb{T})$ can be defined in various ways. For instance, let

$$I_\Phi(f) = \sup_{0 < r < 1} \int_{\mathbb{T}} \Phi(|f(r\zeta)|) |d\zeta|,$$

and define H_Φ [resp., EH_Φ] by the requirement that $I_\Phi(|f|/\lambda) < \infty$ for some [resp., for all] $\lambda > 0$.

From now on, we assume that Φ satisfies Δ_2 -condition so that $H_\Phi = EH_\Phi$.

In view of Lemma 1.1, we may assume (up to equivalent quantities) that $\Phi(t^{1/\alpha})$ is convex, which implies that $\Phi(|f|)$ is subharmonic and hence that $\sup_{0 < r < 1}$ can be replaced with $\lim_{r \uparrow 1}$. The boundary function $f_*(\zeta) = \lim_{r \uparrow 1} f(r\zeta)$ exists because $H_\Phi \subset H^\alpha$, where H^α denotes the standard α -Hardy space, and moreover, we have

$$I_\Phi(f) = \int_{\mathbb{T}} \Phi(|f_*(\zeta)|) |d\zeta|.$$

For the theory of Hardy-Orlicz spaces, we refer to [12, Chapter 10] and [5].

2. Taylor Coefficients of Hardy-Orlicz Functions

In the case where $\Phi(t) = t^p$, $0 < p < 2$, the following result reduces to the famous theorem of Hardy and Littlewood:

Theorem 2.1 ([8]). *If Φ is an Orlicz function such that $\Phi(t)/t^2$ is decreasing in $t > 0$, and $f(z) = \sum a_n z^n$ belongs to H_Φ , then*

$$\sum_{n=1}^{\infty} \frac{\Phi(n|a_n|)}{n^2} \leq CI_\Phi(f). \quad (8)$$

Proof. Let

$$rE_{\Phi}(r, f) = \int_{r\mathbb{D}} \frac{\Phi(|f(z)|)}{|f(z)|^2} |f'(z)|^2 dA(z),$$

where dA is the Lebesgue measure in the plane, $f(z) \neq 0$ for some $z \in \mathbb{D}$, and $f(0) = 0$. Consider the function

$$N(t) = \int_0^t \left(\int_0^s \frac{\Phi(x)}{x} dx \right) \frac{ds}{s} = \int_0^1 \int_0^1 \frac{\Phi(tsx)}{sx} ds dx. \quad (9)$$

This function is of class C^2 on $(0, \infty)$, and therefore, the Hardy-Stein identity

$$2\pi r \frac{d}{dr} I_N(r, f) = \int_{r\mathbb{D}} (N''(|f|) + N'(|f|)/|f|) |f'|^2 dA, \quad (10)$$

holds (see [11]), provided that $|f(z)| \neq 0$ for $|z| = r$. Simplifying the expression under the integral, we conclude that

$$2\pi \frac{d}{dr} I_N(r, f) = E_{\Phi}(r, f), \quad (11)$$

under the above condition. Since the function $I_{\Phi}(r, f)$ is convex of $\log r$ and continuous on $[0, 1)$, we see that it is absolutely continuous on $[0, 1)$, which together with the inequality $\Phi(t)/4 \leq N(t) \leq \Phi(t)/(\alpha^2)$ (valid because $(sx)^2 \Phi(t) \leq \Phi(sxt) \leq (sx)^{\alpha} \Phi(t)$) implies

$$K^{-1} I_{\Phi}(f) \leq \int_0^1 E_{\Phi}(r, f) dr \leq K I_{\Phi}(f), \quad (12)$$

where K is a constant depending only on α . Let

- $Q(t) = \Phi(\sqrt{t})$;
- $\omega(r) = \left(\sum_{n=1}^{\infty} |a_n| r^n \right)^2$;
- $\sigma(r) = \sum_{n=1}^{\infty} n |a_n|^2 r^{2n-1} = \frac{1}{\pi r} \int_{r\mathbb{D}} |f'|^2 dA$.

Since $|f(z)| \leq \omega(r)$ ($|z| < r$) and the function $Q(t)/t$ is decreasing, we have

$$rE_{\Phi}(r, f) \geq \frac{Q(\omega(r))}{\omega(r)} \int_{r\mathbb{D}} |f|^2 dA = \pi r \frac{Q(\omega(r))}{\omega(r)} \sigma(r). \quad (13)$$

It follows that

$$KI_{\Phi}(f) \geq \int_0^1 \frac{Q(\omega(r))}{\omega(r)} \sigma(r) dr. \quad (14)$$

On the other hand,

$$\frac{Q(x)}{x} y \geq Q(y) - Q(x), \quad x > 0, y > 0, \quad (15)$$

because the function $Q(t)/t$ is decreasing. The last two inequalities imply

$$KI_{\Phi}(f) \geq \frac{1}{C} \int_0^1 [Q(C\sigma(r)) - Q(\omega(r))] dr, \quad (16)$$

where $C > 1$ is a constant which will be chosen later on. In order to continue the proof, we need two consequences of [6]:

$$\int_0^1 Q(\sigma(r)) dr \geq K_1 \sum_{n=0}^{\infty} 2^{-n} Q\left(2^n \sum_{k \in I_n} |a_k|^2\right), \quad (17)$$

$$\int_0^1 Q(\omega(r)) dr \leq K_2 \sum_{n=0}^{\infty} 2^{-n} \sum_{k \in I_n} Q\left[\left(\sum_{k \in I_n} |a_k|\right)^2\right], \quad (18)$$

where $I_n = \{k : 2^n \leq k < 2^{n+1}\}$, and K_1, K_2 are positive constants depending only on α . Using this and the inequality

$$\left(\sum_{k \in I_n} |a_k|\right)^2 \leq 2^n \sum_{k \in I_n} |a_k|^2,$$

we get

$$\int_0^1 Q(\omega(r))dr \leq K_3 \int_0^1 Q(\sigma(r)) \leq \frac{1}{2} \int_0^1 Q(C\sigma(r))dr,$$

where C is chosen so that $C^{\alpha/2} \geq 2K_3$. From this and (16) we infer

$$KI_{\Phi}(f) \geq \int_0^1 Q(\sigma(r))dr \asymp \sum_{n=0}^{\infty} 2^{-n} Q\left(2^{-n} \sum_{k \in I_n} k^2 |a_k|^2\right). \quad (19)$$

Since $Q(t)/t$ is decreasing, there is a concave function $Q_1 \prec Q$, so we may assume that Q is concave. Then we have

$$Q\left(2^{-n} \sum_{k \in I_n} k^2 |a_k|^2\right) \geq 2^{-n} \sum_{k \in I_n} Q(k^2 |a_k|^2) = 2^{-n} \sum_{k \in I_n} \Phi(k|a_k|),$$

which along with (19) concludes the proof. \square

Remark 2.2. Inequality (19) can be rewritten as

$$KI_{\Phi}(f) \geq \sum_{n=0}^{\infty} 2^{-n} \Phi\left(2^{n/2} \left(\sum_{k \in I_n} |a_k|^2\right)^{1/2}\right). \quad (20)$$

Theorem 2.3. Let $\Phi \in \Delta[2, \beta]$ for some $\beta \geq 2$. Then

$$I_{\Phi}(f) \leq K \sum_{n=1}^{\infty} \frac{\Phi(n|a_n|)}{n^2}. \quad (21)$$

Proof. With the notation of the proof of Theorem 2.1, we have

$$\frac{Q(x)}{x} y \leq \Phi(y) + \Phi(x).$$

Then proceed in a similar way as in the case of Theorem 2.1. \square

Theorem 2.4. *Let $\Phi \in \Delta(p, q)$ for some $0 < p < q < \infty$. Then*

$$\int_0^1 \Phi\left((1-r)^{-1/q} M_q(r, f)\right) dr \leq KI_\Phi(f). \quad (22)$$

For the proof, we need a consequence of [9, Theorem 1.1].

Lemma 2.5. *Let $F(x, y)$, $0 \leq x \leq 1$, $y \geq 0$, be a nonnegative function such that*

$$\lambda^b \mu^\eta F(\lambda x, \mu y) \leq \lambda^a \mu^\rho F(x, y), \quad (23)$$

where a, b, η, ρ are positive constants independent of $x \in [0, 1]$ and $y \in [0, \infty)$. If $h(r) = \sum_{n=1}^{\infty} c_n r^n$, $0 \leq r < 1$, and $c_n \geq 0$ for all n , then

$$\int_0^1 F[1-r, h(r)](1-r)^{-1} dr \leq \sum_{n=0}^{\infty} F\left(2^{-n}, \sum_{k \in I_n} c_k\right). \quad (24)$$

Proof of Theorem 2.4. Using Blaschke products in the standard way, we reduce the proof to the case $q = 2$. Then $\Phi \in \Delta[\alpha, \beta]$ for some $\beta < 2$. So we have to prove

$$I = \int_0^1 \Phi\left((1-r)^{-1/2} M_2(r, f)\right) dr \leq KI_\Phi(f). \quad (25)$$

The function

$$F(x, y) = \Phi(x^{-1/2} \sqrt{y})x, \quad (26)$$

satisfies (23) with $a = 1 - \beta/2 > 0$, $\alpha = 1 - \alpha/2 > 0$, $\rho = \alpha/2$, $\eta = \beta/2$.

Hence, by Lemma 2.5,

$$\begin{aligned}
 I &= \int_0^1 F\left((1-r)^{-1/2}, \sum_{n=1}^{\infty} |a_n|^2 r^{2n}\right) (1-r)^{-1} dr \\
 &\asymp \sum_{n=0}^{\infty} F\left(2^{n/2}, \sum_{k \in I_n} |a_k|^2\right) \\
 &= \sum_{n=0}^{\infty} 2^{-n} \Phi\left(2^{n/2} \left(\sum_{k \in I_n} |a_k|^2\right)^{1/2}\right).
 \end{aligned}$$

Now the desired result follows from (20). □

As a consequence, we have:

Theorem 2.6. *If $\Phi \in \Delta(\alpha, 1)$, then*

$$KI_{\Phi}(f) \geq \sum_{n=1}^{\infty} n^{-2} \sup_{1 \leq k \leq n} \Phi(k|a_k|), \tag{27}$$

and consequently,

$$\Phi(n|a_n|) = o(n), \quad n \rightarrow \infty. \tag{28}$$

Proof. This can be deduced from the case $q = 1$ of Theorem 2.4 and the inequality

$$M_1(r, f) \geq \sup_{n \geq 0} |a_n| r^n, \quad 0 < r < 1.$$

□

3. Kislyakov-Xu Interpolation Theorem

An operator T acting from H_{Φ} to the class of nonnegative measurable functions on a set is said to be quasilinear if $T(f + g) \leq K(|f| + |g|)$, where K is a positive constant. In the case, where $\Phi(t) = t^s$, $p < s < q$, the following theorem was proved by Kislyakov and Xu [3].

Theorem 3.1. *Let $p < q < \infty$, and let T be an operator defined on H^p with values in the class of μ -measurable functions, where μ is a positive measure. Let T be of weak types (p, p) and (q, q) , i.e., that there exist constants C_1 and C_2 , independent of f , such that*

$$\mu(f, \lambda) := |\{\zeta \in \mathbb{T} : |f_*| > \lambda\}| \leq \frac{C_1}{\lambda} \|f\|_p^p, \quad f \in H^p, \quad (29)$$

$$|\{\zeta \in \mathbb{T} : |f_*| > \lambda\}| \leq \frac{C_2}{\lambda} \|f\|_q^q, \quad f \in H^q, \quad (30)$$

and $\Phi \in \Delta(p, q)$, then T acts a bounded operator from H_Φ into $L_\Phi(\mu)$.

Here $|S|$ denotes the arc-length measure of $S \subset \mathbb{T}$.

Proof. We have the decomposition $f = g_\lambda + h_\lambda$, where $g_\lambda \in H^p$ and $h_\lambda \in H^\infty$ are analytic, and

$$\begin{aligned} \|g_\lambda\|_p^p &\leq A \int_{|f|>\lambda} |f|^p dl, \\ \|h_\lambda\|_q^q &\leq A \int_{|f|\leq\lambda} |f|^q dl + A\lambda^{2q} \int_{|f|>\lambda} |f|^{-q} dl, \end{aligned}$$

where $A = \text{const.}$. This is a consequence of a result of Bourgain [2]; see [10, Lemma 4.1]. Assuming that $T(f + g) \leq Tf + Tg$ and $C_1 = C_2 = 1$, we have

$$\begin{aligned} \mu(Tf, \lambda) &\leq \mu(Tg_\lambda, \lambda/2) + \mu(Th_\lambda, \lambda/2) \\ &\leq A(2/\lambda)^p \int_{|f|>\lambda} |f|^p dl + A(2/\lambda)^q \int_{|f|\leq\lambda} |f|^q dl \\ &\quad + A(2\lambda)^q \int_{|f|>\lambda} |f|^{-q} dl = I_1(\lambda) + I_2(\lambda) + I_3(\lambda). \end{aligned}$$

Now, we use the formula

$$2\pi I_\Phi(f) = \int_0^\infty |\{\zeta \in \mathbb{T} : |f_*| > \lambda\}| d\Phi(\lambda), \quad (31)$$

to get

$$2\pi I_{\Phi}(Tf) \leq \int_0^{\infty} [I_1(\lambda) + I_2(\lambda) + I_3(\lambda)]d\lambda.$$

We have

$$\begin{aligned} \int_0^{\infty} I_1(\lambda)d\lambda &\leq A2^p \int_{\mathbb{T}} |f|^p d\ell \int_0^{|f|} \lambda^{-p} d\Phi(\lambda) \\ &= A2^p \int_{\mathbb{T}} |f|^p \left(\Phi(|f|)|f|^{-p} + p \int_0^{|f|} \Phi(\lambda)\lambda^{-p-1}d\lambda \right) d\ell. \end{aligned}$$

On the other hand, the hypothesis $\Phi \in \Delta(p, q)$ implies that $\Phi(t)/t^{\alpha}$ increases in t for some $\alpha > p$, which gives

$$\int_0^{|f|} \Phi(\lambda)\lambda^{-p-1}d\lambda \leq C\Phi(|f|)|f|^{-p}.$$

Hence,

$$\int_0^{\infty} I_1(\lambda)d\lambda \leq CI_{\Phi}(f).$$

This implies part of the desired result. The rest is proved similarly. \square

Theorem 3.2 (Hardy). *If Φ is an Orlicz function satisfying Δ_2 -condition, then*

$$\int_0^1 \Phi(M_{\infty}(r, f))dr \leq CI_{\Phi}(f), \quad f \in H_{\Phi}. \quad (32)$$

Proof. This is obtained from the “ p -case” of the Hardy theorem by means of the interpolation theorem. \square

Theorem 3.3 (Prawitz). *If f is univalent in \mathbb{D} and Φ as above, then*

$$I_{\Phi}(f) \leq C \int_0^1 \Phi(M_{\infty}(r, f))dr. \quad (33)$$

Proof. Let

$$\Psi(t) = \int_0^t \frac{\Phi(x)}{x} dx \asymp \phi(t). \quad (34)$$

We have

$$\begin{aligned} 2\pi \frac{d}{dr} I_\Psi(r, f) &= \int_{\mathbb{T}} \Psi'(|f(r\zeta)|) \frac{d}{dr} |f(r\zeta)| |d\zeta| \\ &= \int_{\mathbb{T}} \frac{\Phi(|f(r\zeta)|)}{|f(r\zeta)|} \Re \frac{\overline{f(r\zeta)} f'(r\zeta) \zeta}{|f(r\zeta)|} |d\zeta| \\ &= \frac{1}{r} \Im \int_{|\zeta|=r} \frac{\Phi(|f(\zeta)|)}{|f(\zeta)|^2} \overline{f(\zeta)} df(\zeta) \\ &= \frac{1}{r} \Im \int_{\Gamma_r} \frac{\Phi(|w|)}{|w|^2} \overline{w} dw, \end{aligned}$$

where Γ_r is the image of the circle $|\zeta| = r$ under f ; the curve Γ_r is oriented positively. Let $\Omega_{r, R}$, $R > \max_{|z|=r} |f(z)|$, be the domain bounded by Γ_r and the circle $|z| = R$. Applying the Green's formula

$$\int_{\partial\Omega} F(w) dw = 2i \int_{\Omega} \frac{\partial F}{\partial \bar{w}} du dv \quad (w = u + iv), \quad (35)$$

to the case, where $\Omega = \Omega_{r, R}$ and $F(z) = \frac{\Phi(|z|)}{|z|^2} \bar{z}$, we get

$$\begin{aligned} &\int_{\Gamma_r} \frac{\Phi(|w|)}{|w|^2} \overline{w} dw \\ &= \int_{|w|=R} \frac{\Phi(|w|)}{|w|^2} \overline{w} dw - 2i \int_{\Omega} \left(\frac{\Phi(|f(w)|)}{|w|^2} + \frac{\Phi'(|f(w)|)}{2|w|^3} - \frac{\Phi(|f(w)|)}{2|w|^2} \right) du dv. \end{aligned}$$

²We write $A \asymp B$ if there is a constant C such that $B/C \leq A \leq CA$.

Hence

$$\begin{aligned} \Im \int_{\Gamma_r} \frac{\Phi(|w|)}{|w|^2} \bar{w} dw &= \Im \int_{|w|=R} \frac{\Phi(|w|)}{|w|^2} \bar{w} dw \\ &\quad - \int_{\Omega} \left(\frac{\Phi(|f(w)|)}{|w|^2} + \frac{\Phi'(|f(w)|)}{|w|^3} \right) du dv \\ &\leq \Im \int_{|w|=R} \frac{\Phi(|w|)}{|w|^2} \bar{w} dw = \Phi(R). \end{aligned}$$

If $f(0) = 0$, then from the preceding relations it is easy to deduce that

$$I_{\Phi}(f) \leq CI_{\Psi}(f) \leq C \int_0^1 \Phi(M_{\infty}(r, f)) dr.$$

Otherwise, we apply this inequality to the function $f - f(0)$ to complete the proof. □

Combining the last two theorems, we obtain the following generalization of the Hardy-Prawitz theorem:

Theorem 3.4. *A function f univalent in \mathbb{D} belongs to H_p , if and only if*

$$\int_0^1 \Phi(M_{\infty}(r, f)) dr < \infty.$$

References

- [1] S. N. Bernstein, On majorants of finite or quasi-finite growth, Dokl. Akad. Nauk SSSR (NS), (Russian) 65 (1949), 117-120.
- [2] J. Bourgain, New Banach space properties of the disc algebra and H^{∞} , Acta Math. 152(1-2) (1984), 1-48.
- [3] S. V. Kislyakov and Q. Xu, Real interpolation and singular integrals, St. Petersburg. Math. J. 8(4) (1996), 593-615.

- [4] M. A. Krasnosel'skiĭ and Ja. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Translated from the first Russian Edition by Leo F. Boron, P. Noordhoff Ltd., Groningen, 1961.
- [5] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Composition operators on Hardy-Orlicz spaces, *Mem. Amer. Math. Soc.* 974 (2010), i-v + 74.
- [6] M. Mateljević and M. Pavlović, L^p -behavior of power series with positive coefficients and Hardy spaces, *Proc. Amer. Math. Soc.* 87(2) (1983), 309-316.
- [7] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, Volume 1034, Springer-Verlag, Berlin, 1983.
- [8] M. Pavlović, *Geometry of Complex Banach Spaces*, Ph.D. Thesis, Faculty of Mathematics, Belgrade, 1983.
- [9] M. Pavlović, Integrability of vector-valued lacunary series with applications to function spaces, *Ann. Mat. Pura Appl.* 195(2) (2013), 745-762.
- [10] M. Pavlović, *Function Classes on the Unit Disc: An Introduction*, De Gruyter Studies in Mathematics 52, De Gruyter, Berlin/Boston, 2014.
- [11] Ch. Pommerenke, *Univalent Functions, with a Chapter on Quadratic Differentials* by Gerd Jensen, *Studia Mathematica/Mathematische Lehrbücher, Band XXV*, Vandenhoeck & Ruprecht, Göttingen, 1975.
- [12] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, New York etc.: Marcel Dekker, Inc., 1991.

